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Jacobson radical algebras with Gelfand–Kirillov dimension two over countable fields☆

Agata Smoktunowicz

School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, King's Buildings, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom

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Abstract

It is shown that for every countable field K, there is a finitely generated graded Jacobson radical algebra over K of Gelfand–Kirillov dimension two. Examples of finitely generated Jacobson radical algebras of Gelfand–Kirillov dimension two over algebraic extensions of finite fields of characteristic 2 were earlier constructed by Bartholdi [L. Bartholdi, Branch Rings, thinned rings, tree enveloping rings, Israel J. Math. (in press)].

MSC: 16N20; 16P90

0. Introduction

Gelfand–Kirillov dimension is a non-commutative analogue of Krull dimension. In this paper rings with Gelfand–Kirillov dimension two are studied. The results of Small, Stafford and Warfield show that if R is a finitely generated algebra of GK dimension 1, then the Jacobson radical of R is locally nilpotent [7]. By Bergman's Gap theorem there are no algebras with Gelfand–Kirillov dimension strictly between one and two [4]. In [2] Bell constructed examples of prime affine algebras with Gelfand–Kirillov dimension two and a non-zero locally nilpotent Jacobson radical. In the same paper he asked whether the Jacobson radical in algebras with Gelfand–Kirillov dimension two is locally nilpotent. In [1] Bartholdi showed that the answer to this question is in the negative, provided that the base field is an algebraic extension of F_2. However, it is not known whether a similar result holds for other fields, and in particular for fields of characteristic zero. The aim of this paper it is to show that the answer to Bell's question is negative for algebras over countable fields. The main results of this paper are the following.

Theorem 1. *Let K be a countable field. Then there exists a prime finitely generated graded K-algebra of Gelfand–Kirillov dimension two which is Jacobson radical but not nilpotent.*

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E-mail address: A.Smoktunowicz@ed.ac.uk.

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As a corollary the following theorem may be stated.

Theorem 2. *Let K be a countable field. Then there exists a finitely generated prime K-algebra of Gelfand–Kirillov dimension two which is graded nil but not nilpotent.*

However, the following question remains open:

Question 3 (*Jason Bell, [2]*). *Is there a nil algebra with Gelfand–Kirillov dimension two?*

Let K be a field. A graded K-algebra $R = \oplus_{i=1}^{\infty} R_i$ is *graded nil* if all homogeneous elements in R are nil. It is well known that graded Jacobson radical rings are graded nil, but need not be nil. It was asked by Small and Zelmanov whether there is a graded nil algebra which is not Jacobson radical [9]. Another open question is the following:

Question 4. *Are graded nil algebras over uncountable fields nil?*

These questions are also open in the case of algebras with Gelfand–Kirillov dimension two.

Bell investigated the Jacobson radical in rings with Gelfand–Kirillov dimension three. He showed that if K is a countable field then there is a finitely generated K-algebra with Gelfand–Kirillov dimension three with non-locally nilpotent Jacobson radical [2]. A similar result concerning graded algebras with Gelfand–Kirillov dimension two can be derived.

Theorem 5. *Let K be a countable field. Then there exists a finitely generated prime graded K-algebra of Gelfand–Kirillov dimension two which is Jacobson radical.*

For elementary properties of Gelfand–Kirillov dimension we refer the reader to [4].

In what follows K is a countable field and A is the free K-algebra in three non-commuting indeterminates x, y and z. The set of monomials in x, y, z is denoted by M and $M(n)$ denotes the set of monomials of degree n, for each $n \geq 0$. Thus, $M(0) = \{1\}$ and for $n \geq 1$ the elements in $M(n)$ are of the form $x_1 \cdots x_n$, where $x_i \in \{x, y, z\}$. The K-subspace of A spanned by $M(n)$ will be denoted by $H(n)$ and elements of $H(n)$ will be called *homogeneous polynomials of degree n*. Every polynomial $f \in A$ such that $\deg(f) = d$ can be uniquely presented in the form $f = f_0 + f_1 + \cdots + f_d$, where $f_i \in H(i)$. The elements f_i are the *homogeneous components* of f and $\deg(f)$ denotes the degree of the polynomial f. A right ideal I of A is *homogeneous* if for every $f \in I$ all homogeneous components of f are in I. Let V be a linear space over K; then $\dim_K V$ denotes the dimension of V over K. The Gelfand–Kirillov dimension of an algebra R is denoted by $\mathrm{GKdim}(R)$. For elementary properties of Gelfand–Kirillov dimension we refer the reader to [4].

The methods in this paper are quite different from those of Bartholdi and Bell. The general idea of this paper is similar to that of [6,10].

1. Enumerating elements

Let $\overline{A}$ be the subalgebra of A consisting of polynomials with constant term equal to zero. As usual, $\mathbb{N}$ denotes the set of natural numbers.

The aim is to present an algebra with the desired properties as $\overline{A}/E$ for a suitable ideal E.

We start with two results derived from similar results in [6,10].

Lemma 6. *Let K be a countable field, and let $\overline{A}$ be as above. Then there exists a set $Z \subseteq \mathbb{N}$, with all $i \in Z$ being greater than or equal to 5, such that elements of $\overline{A}$ can be enumerated as f_i for $i \in Z$ (that is, $\overline{A} = \{f_i\}_{i \in Z}$) and such that $i > 3^{2t_i+2}(t_i + 1)^2$ for each $i \in Z$, where t_i is the degree of f_i.*

Proof. The field K is countable and the algebra A is finitely generated over K, so the elements of $\overline{A}$ can be enumerated: say $\overline{A} = \{g_1, g_2, \ldots\}$. We now define an increasing function $\theta : \mathbb{N} \longrightarrow \mathbb{N}$ as follows. Set $\theta(1) := \min\{i \in \mathbb{N} \mid i > 4,\ i > 3^{2\deg(g_1)+2}(\deg(g_1 + 1))^2\}$. Suppose that we have defined $\theta : \{1, \ldots, n\} \longrightarrow \mathbb{N}$ such that $\theta(i) > 3^{2\deg(g_i)+2}(\deg(g_i + 1))^2$, for each $i = 1, \ldots, n$. Then set $\theta(n + 1) := \min\{s \mid s > \theta(1), \ldots, \theta(n)$ and $s > 3^{2\deg(g_{n+1})+2}(\deg(g_{n+1} + 1))^2\}$. If we now rename the elements of $\overline{A}$ by setting $f_{\theta(s)} := g_s$ then we have a listing of the elements of $\overline{A}$ with the required properties. $\square$

Given a subset $S \subseteq H(n)$, for some n, let $B_n(S)$ denote the right ideal of A generated by the set $\bigcup_{k=0}^{\infty} M(nk)S$; that is,

$$B_n(S) = \sum_{k=0}^{\infty} M(nk)SA.$$

Lemma 7. *Let $f_m \in \overline{A}$, and let $f_m = p(1) + p(2) + \cdots + p(d)$ where $d = \deg(f_m)$ and $p(i) \in H(i)$ for all $i \leq d$. For each natural number k let*

$$s(k) = \sum_{n=1}^{k} \left(\sum_{0<i_1,i_2,\ldots,i_n \leq d, i_1+i_2+\cdots+i_n=k} p(i_1)p(i_2)\cdots p(i_n) \right).$$

Define $r(k) = \sum_{j=1}^{k} s(j)$. Then for every number $k > d$, $f_m + f_m r(k-d) - r(k-d) + c = e$ where

$$c = \sum_{i=1}^{d} p(i) \sum_{l=k-d}^{k-i} s(l)$$

and

$$e = \sum_{i=k-d+1}^{k} s(i).$$

Proof. Observe that if $l > d$ then

$$s(l) = \sum_{i=1}^{d} p(i)s(l-i).$$

If $l \leq d$ then

$$s(l) = \sum_{i=1}^{l-1} p(i)s(l-i) + p(l).$$

By the above observations

$$\sum_{l=1}^{d} s(l) = \sum_{l=1}^{d} \sum_{i=1}^{l-1} p(i)s(l-i) + \sum_{l=1}^{d} p(l).$$

Similarly, if $k > d$ then

$$\sum_{l=d+1}^{k} s(l) = \sum_{l=d+1}^{k} \sum_{i=1}^{d} p(i)s(l-i).$$

Consequently,

$$\sum_{l=1}^{k} s(l) = \sum_{l=1}^{k} \sum_{i=1}^{\min(d,l-1)} p(i)s(l-i) + \sum_{l=1}^{d} p(l).$$

It follows that

$$\sum_{l=1}^{k} s(l) = \sum_{i=1}^{d} \sum_{l=i+1}^{k} p(i)s(l-i) + \sum_{l=1}^{d} p(l).$$

But $f_m = \sum_{i=1}^{d} p(i)$, so

$$r(k) - f_m = \sum_{i=1}^{d} \sum_{j=i+1}^{k} p(i)s(j-i).$$

Define $l = j - i$. Now

$$r(k) - f_m = \sum_{i=1}^{d} \sum_{l=1}^{k-i} p(i)s(l).$$

Therefore,

$$r(k) - f_m = \sum_{i=1}^{d} p(i) \sum_{l=1}^{k-d} s(l) + c$$

where

$$c = \sum_{i=1}^{d} \sum_{l=k-d}^{k-i} p(i)s(l).$$

Hence, $r(k) - f_m = f_m r(k-d) + c$. We get $f_m + f_m r(k-d) - r(k-d) + c = e$, since $r(k) = r(k-d) + e$. This finishes the proof. $\square$

Lemma 8. *Let $f_m, p(k), s(k), d$ be as in Lemma 7. If $t > l + d$, $l > d$ are natural numbers, then $s(t) \in \sum_{i=0}^{d-1} H(l+i)s(t-l-i)$. Moreover $s(t) \in \sum_{i=0}^{d-1} s(t-l-i)H(l+i)$.*

Proof. We will prove this first assertion, the proof of the second is done by considering the algebra A^{op} with the opposite multiplication. Observe that for every monomial $p(i_1)p(i_2)\cdots p(i_n)$ with $\sum_{j=1}^{n} i_j = t$ there is the number $1 \le m \le n-2$ such that $i_1 + \cdots + i_m < l$ and $i_1 + \cdots + i_{m+1} \ge l$. Hence, $s(k) = \sum_{m=1}^{n-2} c_m$ where

$$c_m = \sum_{i_1,\ldots,i_{m+1}: l > i_1+\cdots+i_m \ge l - i_{m+1}} p(i_1)\cdots p(i_{m+1}) d_{k-(i_1+\cdots+i_{m+1})}$$

where, for each $r \le k$,

$$d_r = \sum_{n=m+2}^{k} \left(\sum_{i_{m+2}, i_{m+3}, \ldots, i_n : i_{m+2}+i_{n+3}+\cdots+i_n = r} p(i_{m+2})\cdots p(i_n) \right),$$

or $c_m = 0$ when the summation runs over the empty set. Observe that $d_{k-(i_1+\cdots+i_{m+1})} = s(k - (i_1 + \cdots + i_{m+1}))$. Notice that, for each m, $c_m \subseteq C(m)$ where

$$C(m) = \sum_{i_1,\ldots,i_{m+1}: l > i_1+\cdots+i_m \ge l + i_{m+1}} H(i_1 + \cdots + i_{m+1}) s(k - (i_1 + \cdots + i_{m+1})),$$

because $p(i) \subseteq H(i)$ for each i. By the assumptions concerning m, we get $l \le (i_1 + i_2 + \cdots + i_{m+1}) < l + d$. Hence, $s(k) \subseteq \sum_{0 \le i < d} H(l+i)s(k-l-i)$, as required. $\square$

Theorem 9. *Let Z, $\{f_i\}_{i \in Z}$ be as in Lemma 6. Fix m in Z, and let $e(m) = 2^{2^{2^{2^m}}}$ and $w_m = 2^{e(m)+2}$. Then there is a two-sided ideal P_m in $\overline{A}$ such that:*

1. *The ideal P_m is generated by homogeneous elements with degrees larger than $10w_m$.*
2. *There is $g_m \in \overline{A}$, such that $f_m - g_m + f_m g_m \in P_m$.*
3. *There is a linear K-space $F_m \subseteq H(2^{e(m)})$ such that $P_m \subseteq B_{w_m}(F_m)$ and $\dim_K(F_m) < m$.*

Proof. Let $s(k), r(k)$ be as in Lemma 7. Let P_m be the two-sided ideal generated by elements of the form $s(20w_m - i)$ where $0 \le i \le d$ and $d = \deg(f_m)$. Set $g(m) = r(20w_m - d)$. By Lemma 7, applied for $k = 20w_m$, we get $f_m - g_m + f_m g_m = e - c$ where $c = \sum_{i=1}^{d} \sum_{l=20w_m-d}^{20w_m-i} p(i)s(l)$ and $e = \sum_{i=20w_m-d+1}^{20w_m} s(i)$. Notice that $c, e \in P_m$, hence $f_m g_m + f_m - g_m = e - c \in P_m$. Observe that the ideal P_m is generated by homogeneous elements with degrees larger than $10w_m$, because $w_m > m > d = \deg(f_m)$ by Lemma 6.

Set F_m as the K-linear space spanned by all elements of the form $H(i)s(u)H(j)$ where $0 \le i, j \le d$ and $i + u + p = 2^{e(m)}$. Observe that $\dim_K H(i) \le 3^i$, for any i, since A is generated by 3 elements. Therefore, $\dim_K F_m \le 3^{2(d+1)}(d+1)^2$, and by Lemma 6, $\dim_K F_m < m$. We will show that $P_m \subseteq B_{w_m}(F_m)$. It suffices to show that for all $0 \le t, t'$ and all $0 \le z \le d$, we have $H(t)s(20w_m - z)H(t') \subseteq B_{w_m}(F_m)$. By the definition of $B(w_m)(F_m)$ it suffices to show that for all $0 \le f < w_m$ and all $0 \le z \le d$, we have $H(f)s(20w_m - z) \subseteq B_{w_m}(F_m)$.

Fix $0 \le f < w_m$, $0 \le z \le d$. Observe now that by Lemma 8 applied for $t = 20w_m - z$, $l = 2w_m - f$ we have $s(20w_m - z) \subseteq \sum_{i=0}^{d-1} H(2w_m - f + i)s(18w_m - i + f - z)$. Therefore, $H(f)s(20w_m - z) \subseteq H(2w_m)\sum_{i=0}^{d-1} H(i)s(18w_m - z - i + f)$.

Now apply the second part Lemma 8 to $t = 18w_m - z - i + f$ and $l = 18w_m - 2^{e(m)} - z + f$. We can apply Lemma 8, since $t - l = 2^{e(m)} - i > d$, and $l > d$. By Lemma 8, $s(18w_m - z - i + f) \subseteq \sum_{g=0}^{d-1} s(18w_m - z - i - l - g + f)H(g)H(l)$. Hence $s(18w_m - z - i + f) \subseteq \sum_{g=0}^{d-1} s(2^{e(m)} - g - i)H(g)H(l)$.

It follows that $H(i)s(18w_m - z - i) \subseteq H(i)s(2^{e(m)} - i - g)H(g)H(l)\bar{A}$, where $0 \le i, g < d$, and $d = \deg f_m$. Therefore, $H(f)s(20w_m - z)H(m) \subseteq H_{2w_m}F_m\bar{A} \subseteq B_{w_m}(F_m)$ as required. $\square$

2. Definition of $U(2^n)$ and $V(2^n)$

Set $S := \{[2^{2^{2^{2i}}} - i - 1, 2^{2^{2^{2i}}} - 1] \mid i = 5, 6, 7, \ldots\}$. Define $e(i) = 2^{2^{2^{2i}}}$. Then $S := \{[e(i) - i - 1, e(i) - 1] \mid i = 5, 6, 7, \ldots\}$.

Theorem 10. *Let Z, F_i be as in Theorem 9. Then there are K-linear subspaces $U(2^n)$ and $V(2^n)$ of $H(2^n)$ such that for all $n > 0$ we have:*

1. $\dim_K V(2^n) = 2$ *if* $n \notin S$.
2. $\dim_K V(2^{e(i)-i-1+j}) = 2^{2^j}$, *for all* $4 < i$ *and all* $0 \le j \le i$.
3. $V(2^n)$ *is generated by monomials.*
4. $F_i \subseteq U(2^{e(i)})$ *for every* $i \in Z$.
5. $V(2^n) + U(2^n) = H(2^n)$ *and* $V(2^n) \cap U(2^n) = 0$.
6. $H(2^n)U(2^n) + U(2^n)H(2^n) \subseteq U(2^{n+1})$.
7. $V(2^{n+1}) \subseteq V(2^n)V(2^n)$.
8. *If* $n \notin S$ *then there are monomials* $m_1, m_2 \in V(2^n)$ *such that* $V(2^n) = Km_1 + Km_2$ *and* $m_2H(2^n) \subseteq U(2^{n+1})$.

Proof. The proof is similar to the proof of Theorem 3 in [6]. We construct the sets $U(2^n)$ and $V(2^n)$ inductively. Set $V(2^0) := V(1) = Kx + Ky$ and $U(2^0) = U(1) := Kz$. Assume that we have defined $V(2^m)$ and $U(2^m)$ for $m \le n$ in such a way that conditions 1–5 hold for all $m \le n$ and conditions 6–8 hold for all $m < n$. Then we define $V(2^{n+1})$ and $U(2^{n+1})$ in the following way. Observe first that since $U(n) \cap V(n) = 0$ then

$$\{U(n)U(n) + U(n)V(n) + V(n)U(n)\} \cap \{V(n)V(n)\} = 0.$$

Our next step is to make the following observation. If $\overline{V}, P \subseteq V(n)V(n)$ and $\overline{V} \cap P = 0$ then

$$\left\{U(n)U(n) + U(n)V(n) + V(n)U(n) + \overline{V}\right\} \cap P = 0.$$

For, suppose that $c = c_1 + c_2 \in P$ with $c_1 \in U(n)U(n) + U(n)V(n) + V(n)U(n)$ and $c_2 \in \overline{V}$. We claim that $c = 0$. Notice that $c \in P$ and $c_2 \in \overline{V}$ implies that $c_1 = c - c_2 \in P + \overline{V} \subseteq V(n)V(n)$. On the other hand, $c - c_2 = c_1 \in U(n)U(n) + U(n)V(n) + V(n)U(n)$. By the above observation, we get $c_1 = 0$ so that $c = c_2 \in \overline{V}$. However, $c \in P$; so that $c \in P \cap \overline{V} = 0$, as required.

Now we will define $V(2^{n+1})$, $U(2^{n+1})$ inductively, in the following way. Consider the three cases

1. $n \in S$ and $n + 1 \in S$.
2. $n \notin S$.
3. $n \in S$ and $n + 1 \notin S$.

Case 1. Suppose that $n \in S$ and $n + 1 \in S$. Then we define $V(2^{n+1}) := V(2^n)V(2^n)$; so condition 7 certainly holds. Notice that $V(2^{n+1})$ is spanned by monomials, since $V(2^n)$ is spanned by monomials; so condition 3 holds. Moreover, $\dim_K V(2^{n+1}) = (\dim_K V(2^n))^2$. Since $n, n+1 \in S$, it follows that $n = 2^i - i - 1 + j$ for some i and some $0 \le j < i$. By the inductive hypothesis, $\dim_K V(2^n) = 2^{2^j}$. Now $\dim_K V(2^{n+1}) = (2^{2^j})^2 = 2^{2^{j+1}}$, as required for condition 2 (condition 1 does not apply in this case). Set $U(2^{n+1}) := U(2^n)H(2^n) + H(2^n)U(2^n)$; so condition 6 certainly holds.

It is now easy to check that condition 5 holds. Finally, observe that since $n+1 \in S$, we have $2^{n+1} \neq 2^{e(i)}$ for every i; hence condition 4 is empty in this case and so holds trivially. Condition 8 holds trivially since $n \in S$.

Case 2. Suppose that $n \notin S$. Then $\dim_K V(2^n) = 2$ and $V(2^n) = Km_1 + Km_2$, for some distinct monomials from $H(2^n)$, by the inductive hypothesis. Set $V(2^{n+1}) := Km_1m_1 + Km_1m_2$. Then $\dim_K V(2^{n+1}) = 2$, as required. Let $\overline{V} = Km_2m_1 + Km_2m_2$. Then $\overline{V} \cap V(2^{n+1}) = 0$ and $\overline{V} + V(2^{n+1}) = V(2^n)V(2^n)$. Set $U(2^{n+1}) := U(2^n)V(2^n) + V(2^n)U(2^n) + U(2^n)U(2^n) + \overline{V}$. We see that $U(2^{n+1}) \cap V(2^{n+1}) = 0$ and $U(2^{n+1}) + V(2^{n+1}) = H(2^{n+1})$. Observe that since $n \notin S$, we get $2^{n+1} \neq 2^{e(i)}$ for every i, and again 4 holds trivially. To show that 8 holds, observe that $m_2H(2^n) = m_2(U(2^n) + V(2^n)) \subseteq U(2^{n+1}) + m_2(Km_1 + Km_2) = U(2^{m+1})$ as required.

Case 3. Suppose that $n \in S$ while $n+1 \notin S$. Then $n = e(i) - 1$ for some $i > 1$, where $e(i) = 2^{2^{2^{2^i}}}$. By the inductive hypothesis $\dim_K V(2^n) = \dim_K V(2^{e(i)-1}) = \dim_K V(2^{e(i)-i-1+i}) = 2^{2^i}$. Now $\dim_K V(2^n)V(2^n) = 2^{2^{i+1}}$.

Assume first that $i \in Z$. We know that F_i has a basis $\{f_1, \dots, f_s\}$ for some $f_1, \dots, f_s \in H(2^{e(i)})$ and $s < i$. Hence $s < 2^{2^{i+1}} - 2$. Write each f_j as $f_j = \overline{f}_j + g_j$ where $\overline{f}_j \in V(2^{e(i)-1})V(2^{e(i)-1})$ and $g_j \in V(2^{e(i)-1})U(2^{e(i)-1}) + U(2^{e(i)-1})U(2^{e(i)-1}) + U(2^{e(i)-1})V(2^{e(i)-1})$. Since $V(2^{e(i)-1}) \cap U(2^{e(i)-1}) = 0$ this decomposition is unique. Let P be a K-linear subspace of $V(2^{e(i)-1})V(2^{e(i)-1})$ such that $\overline{f}_j \in P$ for all $1 \le j \le s$ and $\dim_K P = 2^{2^{i+1}} - 2$.

Since $V(2^{e(i)-1})V(2^{e(i)-1})$ is spanned by monomials and $\dim_K(V(2^{e(i)-1})V(2^{e(i)-1})) = 2^{2^{i+1}}$ while $\dim_K P = 2^{2^{i+1}} - 2$, there are monomials $m_1, m_2 \in V(2^{e(i)-1})V(2^{e(i)-1})$ such that $Km_1 + Km_2 + P = V(2^{e(i)-1})V(2^{e(i)-1})$ and $P \cap (Km_1 + Km_2) = 0$. Now set $V(2^{n+1}) := Km_1 + Km_2$ and $U(2^{m+1}) := U(2^{e(i)-1})V(2^{e(i)-1}) + V(2^{e(i)-1})U(2^{e(i)-1}) + U(2^{e(i)-1})U(2^{e(i)-1}) + P$. Certainly, conditions 1, 3, 5, 6, 7 hold (and 2 does not apply to this case), and condition 5 holds by the observation from the beginning of the proof of this theorem. We claim that condition 4 holds. Indeed, $\overline{f}_j \in P \subseteq U(2^{n+1})$ and $g_j \subseteq U(2^{n+1})$ for every $1 \le j \le s$, so each $f_j \in U(2^{n+1})$. Therefore $F_i \subseteq U(2^{n+1}) = U(2^{e(i)})$, as required.

Finally, to finish Case 3, consider the case that $i \notin Z$. In this case, take any two monomials q_1, q_2 from $V(2^{e(i)-1})V(2^{e(i)-1})$ and set $V(2^{n+1}) := Kq_1 + Kq_2$. Let Q be a K-linear subspace of $V(2^{e(i)-1})V(2^{e(i)-1})$ such that $Kq_1 + Kq_2 + Q = V(2^{e(i)-1})V(2^{e(i)-1})$ and $Q \cap (Kq_1 + Kq_2) = 0$. Now set $V(2^{n+1}) = Kq_1 + Kq_2$, $U(2^{n+1}) = U(2^{e(i)-1})V(2^{e(i)-1}) + V(2^{e(i)-1})U(2^{e(i)-1}) + U(2^{e(i)-1})U(2^{e(i)-1}) + Q$. It is easy to check all conditions, as in previous cases, noting that $2^{n+1} = 2^{e(i)}$ but $i \notin Z$, so that condition 4 holds trivially. Since $n \in S$ condition 8 holds trivially. □

3. The ideal E

The algebra we require will be presented as a factor algebra $\overline{A}/E$ for an ideal E that we now define.

Definition 11. Let $r \in H(n)$ for some n, and let m be the natural number such that $2^m \le n < 2^{m+1}$. We say that $r \in E(n)$ if and only if for all $0 \le j \le 2^{m+2} - n$ we have

$$H(j)rH(2^{m+2} - j - n) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1}).$$

We define $E = E(1) + E(2) + \cdots$.

Theorem 12. *The set E is a two-sided ideal of $\overline{A}$.*

Proof. The proof is the same as the proof of Theorem 5 in [6]. □

4. Definition of $R(j)$, $Q(j)$

In Section 2, the sets $U(*)$ and $V(*)$ were only defined at powers of 2. In this section we define corresponding sets at all other natural numbers j. These are defined in terms of the $U(2^n)$ and $V(2^n)$ for terms occurring in the binary expansion of j.

Let j be a natural number. Write j in binary form as

$$j = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$$

with $0 \le p_0 < p_1 < \cdots < p_n$.

Let $i \notin S$. Then, by Theorem 10(8) there are monomials $m_{1,i}, m_{2,i} \in V(2^i)$ such that $Km_{1,i} + Km_{2,i} = V(2^i)$ and $m_{2,i}H(2^i) \subseteq U(2^{i+1})$. Define $N(2^i) = Km_{1,i}$, $M(2^i) = U(2^i) + Km_{2,i}$ for $i \notin S$. For $i \in S$ set $N(2^i) = V(2^i)$, $M(2^i) = U(2^i)$ and define $m_{2,i} = 0$. Observe that for every i, $N(2^i) \cap M(2^i) = 0$ and $N(2^i) + M(2^i) = H(2^i)$. Define

$$Q(j) := N(2^{p_n})N(2^{p_{n-1}}) \cdots N(2^{p_0}) = \prod_{i=0}^{n} N(2^{p_{n-i}})$$

and set

$$R(j) := \sum_{k=0}^{n} R(j,k),$$

with

$$R(j,0) := H(j - 2^{p_0})M(2^{p_0}) \quad \text{and} \quad R(j,k) = H(m_k)M(2^{p_k})H(t_k)$$

where

$$t_k = \sum_{i=0}^{k-1} 2^{p_i} \quad \text{and} \quad m_k = \sum_{i=k+1}^{n} 2^{p_i}$$

for each $j, k > 0$.

Note that $Q(2^n) = N(2^n)$ and that $R(2^n) = M(2^n)$.

Lemma 13. *Let j be a natural number. Then $R(j) + Q(j) = H(j)$ and $R(j) \cap Q(j) = 0$.*

Proof. Note that $R(j) \subseteq H(j)$ and $Q(j) \subseteq H(j)$ for all j. Since $N(2^{p_i}) + M(2^{p_i}) = H(2^{p_i})$ for all i by Theorem 10, we get $Q(j) + R(j) = H(j)$. Observe that $N(2^{p_i}) \cap M(2^{p_i}) = 0$ for all i by Theorem 10. Therefore $Q(j) \cap R(j) = 0$. $\square$

Lemma 14. *Let j be a natural number, and let $j = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$ be the binary form of j with $0 \le p_0 < p_1 < p_2 < \cdots < p_n$. Let $0 < t < n$ and let $m = 2^{p_t} + 2^{p_{t+1}} + \cdots + 2^{p_n}$ and $m' = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_{t-1}}$. Then $R(j) = R(m)H(m') + H(m)R(m')$.*

Proof. The proof is very similar to the proof of Lemma 7 in [6]. Notice that $m' + m = j$. Let $R(j) = \sum_{i=0}^{n} R(j,k)$ be as in the definition above. Then

$$R(j,k) = H(m_k)M(2^{p_k})H(l_k)$$

where

$$l_k = \sum_{i=0}^{k-1} 2^{p_i} \quad \text{and} \quad m_k = \sum_{i=k+1}^{n} 2^{p_i}.$$

Suppose that $k < t$; so that $m_k \ge m$. Then $R(j,k) = H(m)H(m_k - m)M(2^{p_k})H(l_k)$. Observe now that $m' = \sum_{i=0}^{t-1} 2^{p_i}$ is the binary form of $m' = j - m$. Therefore $R(m',k) = H(m_k - m)M(2^{p_k})H(l_k)$ for $k < t$. Hence $R(j,k) = H(m)R(m',k)$ for $k < t$, and consequently

$$\sum_{i=0}^{t-1} R(j,k) = H(m)R(m').$$

Now suppose that $k \ge t$; so that $l_k \ge m'$. Then

$$R(j,k) = H(m_k)M(2^{p_k})H(l_k - m')H(m'),$$

and, arguing as above, $R(j,k) = R(m, k-t)H(m')$. Therefore,

$$\sum_{i=t}^{n} R(j,k) = R(m)H(m').$$

The result follows. $\square$

Theorem 15. *For all natural numbers j, t we have*

$$R(j)H(t) \subseteq R(j+t).$$

Moreover for all integers $p \geq 0$, $0 < t < 2^{p+1}$,

$$R(2^{p+1} - t)H(t) \subseteq U(2^{p+1}).$$

Proof. It is sufficient to show that for every $0 < j$, $0 \leq p$, we have $R(j)H(1) \subseteq R(j+1)$ and $R(2^{p+1} - 1)H(1) \subseteq U(2^{p+1})$.

First, consider the case where $j = 2^{p+1} - 1$ for some $p \geq 0$. Then $j = 2^0 + 2^1 + 2^2 + \cdots + 2^p$. Consequently $R(j) = \sum_{k=0}^{p} R(j,k)$, where

$$R(j,k) = H(2^{p+1} - 2^{k+1})M(2^k)H(2^k - 1).$$

Notice that $R(j+1) = R(2^{p+1}) = M(2^{p+1})$. Hence $U(2^{p+1}) \subseteq R(j+1)$. Therefore, it suffices to show that $R(j,k)H(1) \subseteq U(2^{p+1})$, for every $k \geq 0$. Notice that

$$R(j,k)H(1) = H(2^{p+1} - 2^{k+1})M(2^k)H(2^k - 1)H(1) = H(2^{p+1} - 2^{k+1})(U(2^k) + Km_{2,k})H(2^k).$$

By Theorem 10(8) and Theorem 10(6),

$$R(j,k)H(1) \subseteq H(2^{p+1} - 2^{k+1})U(2^{k+1}).$$

Since $H(2^t)U(2^t) \subseteq U(2^{t+1})$, again by Theorem 10(6), we obtain

$$H(2^{p+1} - 2^t)U(2^t) \subseteq H(2^{p+1} - 2^{t+1})U(2^{t+1}).$$

Applying this observation several times for $t = k+1, t = k+2, \ldots, t = p$, we get that $R(j,k)H(1) \subseteq U(2^{p+1})$, as required.

Next, assume that $j \neq 2^{p+1} - 1$ for all p. Write j in binary form: $j = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$ for some $0 \leq p_0 < p_1 < p_2 < \cdots < p_n$.

First, assume that $p_0 \neq 0$. Then $j + 1 = 2^0 + 2^{p_0} + 2^{p_1} + \cdots + 2^{p_n}$ is the binary form of $j+1$. Let $R(j) = \sum_{i=0}^{n} R(j,k)$ and $R(j+1) = \sum_{i=0}^{n+1} R(j+1,k)$ be as in the definition. Now we see that $R(j,k)H(1) \subseteq R(j+1,k+1)$. Therefore $R(j)H(1) \subseteq R(j+1)$ as required.

Next, assume that $p_0 = 0$, and let t be minimal such that $p_t - p_{t-1} > 1$. Then $p_i = i$ for all $0 \leq i \leq t-1$ and $p_t > t$. Therefore $j = 2^t - 1 + \sum_{i=t}^{n} 2^{p_i}$. By using the previous lemma, observe that $R(j) = R(m)H(m') + H(m)R(m')$ where $m' = \sum_{i=0}^{t-1} 2^{p_i} = 2^t - 1$ and $m = \sum_{i=t}^{n} 2^{p_i}$. Thus,

$$R(j)H(1) = R(m)H(m')H(1) + H(m)R(m')H(1).$$

Since $m' = 2^t - 1$, we get $R(m')H(1) \subseteq U(2^t) \subseteq R(m'+1) = R(2^t)$, by the first part of the proof. Therefore

$$R(j)H(1) \subseteq R(m)H(m'+1) + H(m)R(m'+1).$$

Observe that the binary form of $j+1$ is $j + 1 = 2^t + \sum_{i=t}^{n} 2^{p^i}$. Recall that $m = \sum_{i=t}^{n} 2^{p^i}$ and that $2^t = m' + 1$. Now by Lemma 14, we get

$$R(j+1) = R(m)H(m'+1) + H(m)R(m'+1).$$

Consequently $R(j)H(1) \subseteq R(j+1)$, and the lemma follows. $\square$

5. Dimensions of linear spaces

Theorem 16. *Let n be a natural number. Then* $\mathrm{Dim}_K V(2^n) < 3\log(n+2)$.

Proof. By the definition if $n \notin S$ then $\mathrm{Dim}_K V(2^n) = 2 < 3\log(n+2)$. If $n \in S$ then $n = 2^{2^{2^{2^i}}} - i - 1 + j$ for some $i \geq 5$, and for some $j \leq i$. By Theorem 10(2) $\dim_K V(2^{2^{2^{2^{2^i}}}-i-1+j}) \leq 2^{2^i}$, for all $i \geq 5$. Note that $2^{2^{2^{2^i}}} - i - 1 > 2^{2^{2^i}}$ for all $i = 1, 2, \ldots$ (proof by induction on i). Hence if $n \in S$ and $n = 2^{2^{2^{2^i}}} - i - 1 + j$ then $n > 2^{2^{2^i}}$. Now $\mathrm{Dim}_K V(2^n) \leq 2^{2^i}$, yields $\mathrm{Dim}_K V(2^n) < 3\log(n)$. $\square$

Theorem 17. *Let n be a natural number. Then* $\mathrm{Dim}_K\, Q(n) < \log(n+2)$.

Proof. If $n \le 2$ then $\mathrm{Dim}_K\, Q(n) = 1 < \log 3 \le \log(n+2)$. Assume now that $n > 2$. Let $n = 2^{p_0} + 2^{p_1} + \cdots + 2^{p_m}$ with $0 \le p_0 < p_1 < p_2 < \cdots < p_m$. Then $Q(n) = N(2^{p_n})N(2^{p_{n-1}})\ldots N(2^{p_0})$. Recall that $\dim_K N(2^k) = 1$, provided that $k \notin S$.

Therefore $\dim_K\, Q(n) \le \prod_{i=0}^{\lfloor \log(n)\rfloor} \dim_K\, N(2^i)$, where $\lfloor \log(n)\rfloor$ is the largest integer not exceeding $\log(n)$. Recall that $S = \{[e(i)-i-1, e(i)-1] \mid i = 5, 6, \ldots\}$, where $e(i) = 2^{2^{2^{2^i}}}$.

Now let $c_i = \prod_{t=e(i)-i-1}^{e(i)-1} \dim_K\, N(2^t)$. We see that $c_i = \prod_{j=0}^{i} \dim_K\, N(2^{e(i)-i-1+j}) = \prod_{j=0}^{i} 2^{2^j} < 2^{2^{i+1}}$, by Theorem 10(2). Since $\dim_K N(2^k) = 1$ if $k \notin S$ we have $\dim_K\, Q(n) \le \prod_{i\in S, i=0,\ldots\lfloor\log(n)\rfloor} \dim_K\, N(2^i)$. Let q be the maximal number such that $e(i) - q - 1 \le \lfloor \log(n)\rfloor$. Then $\dim_K(Q_n) \le \prod_{i=0}^{q} c_i \le \prod_{i=0}^{q} 2^{2^{q+1}} \le 2^{2^{q+2}}$.

Observe that $2^{2^{2^{q+2}}} < 2^{2^{2^{2^q}}} - (q+1)$ (proof by induction). On the other hand $2^{2^{2^{2^q}}} - (q+1) = e(i) - q - 1 \le n$. Therefore, $2^{2^{q+2}} < \log(n)$. Therefore $\dim_K(Q(n)) < \log(n)$ for $n \ge 2$. □

Theorem 18. *Let n, m be natural numbers and $2^m \le n < 2^{m+1}$. Then there are K-linear subspaces $W(n) \subseteq H(n)$, $Z(n) \subseteq H(n)$ such that*

1. $W(n) + Z(n) = H(n)$, $W(n) \cap Z(n) = 0$.
2. $\dim_K(Z(n)) < 3(\log(n)+3)^2$.
3. $H(2^{m+1}-n)W(n) \subseteq U(2^{m+1})$.
4. $H(2^k - n)W(n) \subseteq U(2^k)$ *for each* $k \ge m+1$.
5. *If* $n = 2^m$ *for some number* $m \ge 0$, *then* $W(n) = U(n) = U(2^m)$.

Proof. If $n = 2^m$ we put $W(n) = U(n) = U(2^m)$ and $Z(n) = V(n)$; this proves property 5. Notice that in this case, property 2 holds by Theorem 16 and properties 1, 3, 4 hold by Theorem 10(5) and Theorem 10(6). Assume now that $2^m < n < 2^{m+1}$ for some natural number m. Now, we will prove properties 1, 2, 3. By Theorem 15, we have $R(2^{m+1}-n)H(n) \subseteq U(2^{m+1})$. Therefore, if $S \subseteq H(n)$ and $Q(2^{m+1}-n)S \subseteq U(2^{m+1})$, then $H(2^{m+1}-n)S \subseteq U(2^{m+1})$. By Theorem 17 there are $r_i \subseteq H(2^{m+1}-n)$ such that $Q(2^{m+1}-n) = \sum_{i=1}^{p} Kr_i$ where $p < \log(2^{m+1}-n+2) < m+3$. Fix one monomial r_g. Let $c_1, \ldots, c_d \in H(n)$ and $d > 3\log(m+3)$. Then $d > \dim_K(V(2^{m+1}))$, by Theorem 16. Note that $r_g c_j \subseteq H(2^{m+1}) = U(2^{m+1}) + V(2^{m+1})$, by Theorem 10(5). Since $d > \dim_K(V(2^{m+1}))$ then there are $\alpha_1, \ldots, \alpha_d$ (not all equal to zero) such that $\sum_{i=1}^{d} \alpha_i r_g c_i \subseteq U(2^{m+1})$. Therefore, if $c_1, \ldots, c_k \in H(n)$ and $k > 3(m+3)\log(m+3)$ then there are $\alpha_1, \ldots, \alpha_k$ such that $r_j \sum_{i=1}^{k} \alpha_k c_i \in U(2^{m+1})$ for all $1 \le j \le p$. We conclude that if $S \subseteq H(n)$ is a K-linear space and $\dim_K S > 3(m+3)\log(m+3)$ then there is $c \in S$, $c \ne 0$, such that $Q(2^{m+1}-n)c \subseteq U(2^{m+1})$, and consequently $H(2^{m+1}-n)c \subseteq U(2^{m+1})$.

Let $W(n) \subset H(n)$ be the K-linear space maximal with the property $H(2^{m+1}-n)W(n) \subseteq U(2^{m+1})$. This linear space is unique, because $U(2^{m+1})$ is a linear space. Let $Z(n) \subseteq H(n)$ be a K-linear space maximal with the property $Z(n) \cap W(n) = 0$. Then $H(n) = Z(n) + W(n)$. Notice that, by the first part of the proof, we get that $\mathrm{Dim}_K Z(n) \le 3(m+3)\log(m+3) < 3(m+3)^2 \le 3(\log(n)+3)^2$ (since $Z(n) \cap W(n) = 0$). Now, we will show that property 4 holds. We proceed by induction on k. If $k = m+1$ then property 4 holds. Assume that it holds for some k; then $H(2^k - n)W(n) \subseteq U(2^k)$. Now, multiplying by $H(2^k)$ from the left, we get $H(2^{k+1}-n)W(n) \subseteq H(2^k)U(2^k) \subseteq U(2^{k+1})$, by Theorem 10(6), as required. □

6. Estimation of the Gelfand–Kirillov dimension

In order to estimate the Gelfand–Kirillov dimension of $\overline{A}/E$, we need to recognize when certain homogeneous elements are in E. The next theorem provides a sufficient condition for this to happen.

Theorem 19. *Let n be a natural number, and let m be the natural number such that $2^m \le n < 2^{m+1}$. Suppose that for every $1 \le j \le n-1$, $r \subseteq W(j)H(n-j) + H(j)R(n-j)$, and $r \subseteq W(n)$ and $r \subseteq R(n)$. Then $r \in E$.*

Proof. By the definition of E we have to show that for all $0 \le j \le 2^{m+2} - n$ we have

$$H(j)rH(2^{m+2}-j-n) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1}).$$

Consider the four possibilities:

1. $j + n > 2^{m+1}$ and $j \le 2^{m+1}$.
2. $j > 2^{m+1}$.
3. $j + n < 2^{m+1}$.
4. $j + n = 2^{m+1}$.

Case 1. If $j = 2^{m+1}$ then by the assumptions of our theorem $r \subseteq R(n)$ and $0 < 2^{m+2} - j - n < 2^{m+1}$; hence $H(j)rH(2^{m+2} - j - n) \subseteq H(2^{m+1})R(n)H(2^{m+2} - n - j) \subseteq H(2^{m+1})U(2^{m+1})$ by Theorem 15. If $j < 2^{m+1}$, then set $t = 2^{m+1} - j$. Then $0 < t < n$ by Assumption 1. The assumptions of our theorem yield $r \subseteq W(t)H(n-t) + H(t)R(n-t)$. Hence, $H(j)rH(2^{m+2} - j - n) \subseteq H(j)W(t)H(2^{m+1}) + H(j + t)R(n-t)H(2^{m+2} - j - n)$. By Theorem 18, we have $H(j)W(t) \subseteq U(2^{m+1})$. Observe that since $j < 2^{m+1}$ and $n < 2^{m+1}$, we get $0 < 2^{m+2} - j - n < 2^{m+1}$. By Theorem 15, $R(n-t)H(2^{m+2} - j - n) \subseteq U(2^{n+1})$. Hence, $H(j)rH(2^{m+2} - j - n) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1})$, as required.

Case 2. Suppose that $j > 2^{m+1}$. Then $j = 2^{m+1} + b$ for some $b > 0$. Since $j + n \le 2^{m+2}$, we have $b + n \le 2^{m+1}$. If $b + n = 2^{m+1}$ then by the assumption $r \subseteq W(n)$ and by Theorem 18, we get $H(b)r \in H(b)W(n) \subseteq U(2^{m+1})$. Hence $H(j)r = H(2^{m+1})H(b)r \subseteq H(2^{m+1})U(2^{m+1})$ as required.

Assume now that $2^{m+1} > b + n$. Then, $b < 2^m$, since $n \ge 2^m$. Take $t = 2^m - b$. Then $0 < t < n$, since $b > 0$. Hence, by the assumption, $r \in W(t)H(n-t) + H(t)R(n-t)$. Consequently, $H(b)rH(2^{m+1} - n - b) \subseteq H(b)W(t)H(2^m) + H(b+t)R(n-t)H(2^m - n + t)$. Note that $b + t = 2^m$, so $H(b)W(t)H(2^m) \subseteq U(2^m)H(2^m)$, by Theorem 18. Notice that $0 < 2^m - n + t = 2^{m+1} - n - b < 2^m$, because $n \ge 2^m$, $b > 0$. Theorem 15 gives $R(n-t)H(2^{m+1}-n-b) \subseteq U(2^m)$. Consequently, $H(b)rH(2^{m+2}-j-n) \subseteq U(2^{m+1})$. Now $H(j)rH(2^{m+2}-j-n) \subseteq H(2^{m+1})U(2^{m+1})$, since $j = 2^{m+1} + b$.

Case 3. If $j = 0$ then $H(j)rH(2^{m+2} - n) \subseteq rH(2^{m+1} - n)H(2^{m+1})$. Assumptions $r \in R(n)$, $0 < n < 2^{m+1}$ and Theorem 15 yields $rH(2^{m+1} - n) \subseteq U(2^{m+1})$. Hence $H(j)rH(2^{m+2} - n) \subseteq U(2^{m+1})H(2^{m+1})$, as required.

Suppose that $j + n < 2^{m+1}$ and $j > 0$. Then $j < 2^m$, since $n \ge 2^m$. Set $t := 2^m - j$. Then $0 < t < 2^m$. By the assumptions of the theorem, $r \in W(t)H(n-t) + H(t)R(n-t)$. Note that $j + t = 2^m$. Note that

$$H(j)rH(2^{m+2} - n - j) = H(j)rH(2^{m+1} - n - j)H(2^{m+1}).$$

Observe that $H(j)rH(2^{m+1} - n - j) \subseteq H(j)W(t)H(2^m) + H(2^m)R(n-t)H(2^{m+1} - n - j)$.

By Theorem 18, $H(j)W(t) \subseteq U(2^m)$, and therefore $H(j)W(t)H(2^m) \subseteq U(2^m)H(2^m) \subseteq U(2^{m+1})$. Observe that $n - t + 2^{m+1} - n - j = 2^m$. By the assumption about Case 3, $0 < n - t < 2^m$. Theorem 15 gives $R(n-t)H(2^{m+1} - n - j) \subseteq U(2^m)$. Hence $H(2^m)R(n-t)H(2^{m+1} - n - j)H(2^{m+1}) \subseteq H(2^m)U(2^m)H(2^{m+1}) \subseteq U(2^{m+1})H(2^{m+1})$. Consequently $H(j)rH(2^{m+2} - n - j) \subseteq U(2^{m+1})H(2^{m+1})$.

Case 4. By the assumption, $r \in W(n)$. By Theorem 18, $H(j)r \subseteq H(j)W(n) \subseteq U(2^{m+1})$. Consequently, $H(j)rH(2^{m+2} - j - n) \subseteq U(2^{m+1})H(2^{m+1})$. This finishes the proof. $\square$

After all this preparation, we can now estimate the Gelfand–Kirillov dimension of our factor algebra.

Theorem 20. $\mathrm{GKdim}(\overline{A}/E) \le 2$.

Proof. Let $0 \le j \le n$. Define $W(0) = R(0) = 0$, $H(0) = Z(0) = Q(0) = K$, where K is the base field. Note that $W(j) + Z(j) = H(j)$ and $R(n-j) + Q(n-j) = H(n-j)$. It follows that

$$H(n) = H(j)H(n-j) = Z(j)Q(n-j) + \{W(j)H(n-j) + H(j)R(n-j)\}.$$

Notice that $\dim_K Z(n) < 3(\log(n) + 3)^2$ and $\dim_K Q(n) \le \log(n+2)$, by Theorems 17 and 18.

Thus, $\dim_K \frac{H(n)}{W(j)H(n-j)+H(j)R(n-j)} \le \dim W(j)Q(n-j) \le 3\log(n+2)(\log(n)+3)^2$.

Let

$$\theta : H(n) \longrightarrow \bigoplus_{j=0}^{n} \frac{H(n)}{\{W(j)H(n-j) + H(j)R(n-j)\}}$$

be the natural map. Then

$$\ker(\theta) = \{r \in H(n) \mid r \in W(j)H(n-j) + H(j)R(n-j) \text{ for each } 0 \le j \le n\} \subseteq E(n)$$

by Theorem 19. Thus,

$$\dim\left(\frac{H(n)}{E(n)}\right) \leq \dim\left(\frac{H(n)}{\ker(\theta)}\right) \leq 3(n+1)\log(n+2)(\log(n)+3)^2.$$

Consequently, $\mathrm{GKdim}(\overline{A}/E) \leq 2$. $\square$

7. $\overline{A}/E$ is Jacobson radical but not nilpotent

We show that $\overline{A}/E$ is Jacobson radical by showing that the ideals P_j defined in Section 1 belong to E. In order to see that $\overline{A}/E$ is not nilpotent we show that the K-subspaces $V(2^n)$ are not contained in E.

Lemma 21. *Let Z, $\{P_j\}_{j\in Z}$, $\{F_j\}_{j\in Z}$ be as in Theorem 10. Fix any $i \in Z$ and suppose that $m+2 > 2^{2^{2^{2^i}}}$. Then $B_{w_i}(F_i) \cap H(2^{m+2}) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1})$.*

Proof. The proof is very similar to the proof of Lemma 12 in [6]. Let $e(i) = 2^{2^{2^{2^i}}}$. We know that $F_i \subseteq H(2^{e(i)})$ and $w_i = 4r_i$ where $r_i = 2^{e(i)}$. Set $w := w_i$ and $r := r_i$. By the assumptions of this lemma, $2^{m+1} \geq 2^{e(i)}$. Observe that $B_w(F_i) \subseteq B_r(F_i)$, since $w = 4r$. Also, $B_r(F_i) \subseteq B_r(U(r))$, since $F_i \subseteq U(2^{e(i)}) = U(r)$ by Theorem 10(4). Consequently

$$B_w(F_i) \subseteq B_r(U(r)).$$

Therefore, it is sufficient to show that

$$B_r(U(r)) \cap H(2^{m+2}) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1})$$

for all m such that $2^{m+1} \geq 2^{e(i)} = r$. We will proceed by induction on m. If $m+1 = e(i)$ then $2^{m+2} = 2r$; so $B_r(U(r)) \cap H(2r) = U(r)H(r) + H(r)U(r)$, by the definition of $B_r(U(r))$, and the fact that $U(r) \subseteq H(r)$.

Suppose now that the result holds for some m, with $2^{m+1} \geq 2^{e(i)} = r$. We will prove that the result holds for $m+1$. We have to show that

$$B_r(U(r)) \cap H(2^{m+3}) \subseteq U(2^{m+2})H(2^{m+2}) + H(2^{m+2})U(2^{m+2}).$$

Observe that, since r divides 2^{m+2}, we obtain

$$B_r(U(r)) \cap H(2^{m+3}) = \{B_r(U(r)) \cap H(2^{m+2})\}H(2^{m+2}) + H(2^{m+2})\{B_r(U(r)) \cap H(2^{m+2})\},$$

by the definition of $B_r(U(r))$. By the induction assumption

$$B_r(U(r)) \cap H(2^{m+2}) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1}) \subseteq U(2^{m+2})$$

by Theorem 5(6). Hence, $B_r(U(r)) \cap H(2^{m+3}) \subseteq U(2^{m+2})H(2^{m+2}) + H(2^{m+2})U(2^{m+2})$ and the result follows. $\square$

Theorem 22. *Let $i \in Z$ and let P_i, f_i, w_i be as in Lemma 6 and Theorem 9, so in particular $P_i \subseteq B_{w_i}(F_i)$. Then $P_i \subseteq E$.*

Proof. Let $r \in P_i$. According to Theorem 9(1), $r = \sum_{p=10w_i}^{s} r_p$ for some $r_p \in H(p)$, and some s. Fix n, with $10w_i \leq n \leq s$. It is sufficient to show that $r_n \in E$. Let m be the natural number such that $2^m \leq n < 2^{m+1}$. Let $e(i) = 2^{2^{2^{2^i}}}$. Note that $10w_i = 2^{e(i)}40$; so $2^{e(i)}40 \leq n < 2^{m+1}$. Hence $m+1 > e(i)$. In order to show that $r_n \in E$, we have to show that

$$H(j)r_nH(2^{m+2}-n-j) \subseteq U(2^{m+1})H(2^{m+1}) + H(2^{m+1})U(2^{m+1}),$$

for every $0 \leq j \leq 2^{m+2}-n$.

Now $r \in P_i$ yields $H(j)rH(2^{m+2}-n-j) \in P_i$. Consequently, $H(j)rH(2^{m+2}-n-j) \subseteq B_{w_i}(F_i)$, by Theorem 9; so

$$H(j)\left(\sum_{p=10w_i}^{s} r_p\right)H(2^{m+2}-n-j) \subseteq B_{w_i}(F_i).$$

It follows that $H(j)r_p H(2^{m+2}-n-j) \subseteq B_{w_i}(F_i)$, for every p with $10w_i \leq p \leq s$, since $B_{w_i}(F_i)$ is homogeneous and $r_p \in H(p)$ for every p.

In particular,

$$H(j)r_n H(2^{m+2} - n - j) \subseteq B_{w_i}(F_i).$$

Now, since $H(j)r_n H(2^{m+2} - n - j) \subseteq H(2^{m+2})$ and $m + 2 > e(i)$, we have

$$\begin{aligned} H(j)r_n H(2^{m+2} - n - j) &\subseteq B_{w_i}(F_i) \cap H(2^{m+2}) \\ &\subseteq H(2^{m+1})U(2^{m+1}) + U(2^{m+1})H(2^{m+1}), \end{aligned}$$

by Lemma 21, and this completes the proof. □

The next two results are now immediate.

Corollary 23. *Let Z, $\{f_i\}_{i\in Z}$, $\{P_i\}_{i\in Z}$ be as in Lemma 6 and Theorem 9. Let J be the two-sided ideal in A generated by all ideals P_i where $i \in Z$, i.e., $J = \sum_{i\in Z} P_i$. Then $J \subseteq E$.*

Theorem 24. *The algebra $\overline{A}/E$ is a Jacobson radical algebra.*

Proof. We use the fact that a ring in which every element is right quasiregular is Jacobson radical [5]. Let $r \in \overline{A}$. According to Lemma 6, $r = -f_m$ for some $m \in Z$. Theorem 9 gives $f_m g_m - g_m + f_m \in P_m$, for some $g_m \in \bar{A}$. Notice that $P_m \subseteq E$, by Corollary 23. Define $r' = g_m$. Then $rr' + r + r' = -f_m g_m - f_m + g_m \in E$. Hence $rr' + r + r' = 0$ in $\overline{A}/E$, so r is right quasiregular. Therefore, $\overline{A}/E$ is Jacobson radical. □

Finally, we show that $\overline{A}/E$ is not nilpotent.

Theorem 25. *The algebra $\overline{A}/E$ is not nilpotent.*

Proof. The proof is identical to the proof of Theorem 16 in [6]. □

In conclusion, we have proved:

Theorem 26. *Let K be a countable field. Then the K-algebra $\overline{A}/E$:*

1. *is generated by three elements x, y, z;*
2. *is Jacobson radical, but not nilpotent;*
3. *has Gelfand–Kirillov dimension 2;*
4. *is graded by the set of natural numbers, and the elements x, y, z have gradation 1.*

Proof. To prove 1, observe that since $\overline{A}$ is generated by elements x, y, z, it follows that $\overline{A}/E$ is also generated by elements x, y, z. Since E is homogeneous and x, y, z have gradation 1, we get 4. Note that 2 follows from Theorems 24 and 25. It remains to show that the Gelfand–Kirillov dimension of $\overline{A}/E$ equals 2. By Theorem 20, GKDim $\overline{A}/E \leq 2$. By the Small–Stafford–Warfield theorem a finitely generated algebra with GKDim 1 satisfies a polynomial identity. By the Rasmyslov–Kemer–Brown theorem [3], an affine Jacobson radical algebra satisfying a polynomial identity is nilpotent. Hence $\overline{A}/E$ cannot have GK dimension 1 [8,11]. By Bergman's Gap theorem there are no algebras with Gelfand–Kirillov dimension strictly between 1 and 2 [4]. Hence, GKDim $\overline{A}/E = 2$. □

Proof of Theorem 1. Let $R = \overline{A}/E$ be as in Theorem 26. By the Small–Stafford–Warfield theorem a Jacobson radical algebra with GK dimension not exceeding one is nilpotent [11]. Let $L(R)$ be the locally nilpotent radical of R. Observe that $R/L(R)$ is not nilpotent, because R is not nilpotent. By Bergman's Gap theorem there are no algebras with GK dimension strictly between one and two. Therefore, $R/L(R)$ has Gelfand–Kirillov dimension two. Let $R' = R/L(R)$ and let $P(R')$ be a prime radical of R. Then $P(R') = 0$ since R' has no nilpotent ideals. Recall that $P(R')$ is the intersection of all prime ideals of R'. It follows that there is a homogeneous prime ideal $P \neq R'$ in R'. Observe that R'/P is prime, and so not nilpotent. Consequently, R'/P is Jacobson radical, and has Gelfand–Kirillov dimension two. □

Proof of Theorem 2. It follows from Theorem 1, because graded Jacobson radical rings are graded nil. □

Proof of Theorem 5. It follows from Theorem 1. □

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